Boundedness for Threefolds in $\mathbb{P}^6$ Containing a Smooth Ruled Surface as Hyperplane Section

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ABSTRACT

Let $X \subset \mathbb{P}^6$ be a smooth irreducible projective threefold, and d its degree. In this paper we prove that there exists a constant β such that for all X containing a smooth ruled surface as hyperplane section and not contained in a fourfold of degree less than or equal to 15, $d \leq \beta$. Under some more restrictive hypothesis we prove an analogous result for threefolds containing a smooth ruled surface as hyperplane section and contained in a fourfold of degree less than or equal to 15.

Key words: threefolds, low codimensional subvarieties, ruled surface as hyperplane section, bounded degree.

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Introduction

Let $\mathcal{S}(\mathbb{P}^N)$ be the set of all irreducible smooth projective n dimensional varieties of the complex projective space $\mathbb{P}^N$. As in [5] we say that a subset $\mathcal{S} \subset \mathcal{S}(\mathbb{P}^N)$ is bounded, or equivalently that varieties in $\mathcal{S}$ are bounded, if

$$\sup\{\deg(X) : X \in \mathcal{S}\} < +\infty.$$

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A famous conjecture of Hartshorne and Lichtenbaum states that n dimensional non general type varieties of $\mathbb{P}^N$, with $N \geq 2n$, are bounded. In [7] the conjecture is proved for $n = 2$, $N = 4$, in [2] for $n = 3$, $N = 5$, and in [13] it is proved in the range $n \geq \frac{N+2}{2}$. As far as we know, nothing is known for $n = 3$, $N = 6$.

In this paper we will discuss boundedness for smooth threefolds in $\mathbb{P}^6$ containing a smooth ruled surface as hyperplane section, or *threefolds with ruled surface section* for short. The reason for restricting to this class of varieties is that in this case we have a, at least coarse, classification (see [1, p. 205, table 7.3]. By analogy with [7] (see also [5]), we first bound the degree of threefolds with ruled surface section not contained in a fourfold of fixed degree less than or equal to 15. This is the content of our main result, Theorem 2.1. Section 2 is devoted to the proof of this Theorem. Next we face boundedness for threefolds contained in a fourfold of fixed degree. We are able to prove only a partial result in this direction, this is discussed in section 3.

1. Preliminaries

We work over the complex numbers.

Notation 1.1. Denote by $X \subset \mathbb{P}^6$ a smooth irreducible projective variety, $\dim(X) = 3$, and H the class of an hyperplane section. By $d = H^3$ we will denote the degree of X, by g its sectional genus and by K_X the class of its canonical bundle. If T_X denotes the tangent bundle of X we will adopt the following notation for its Chern classes $c_i(X) := c_i(T_X)$.

In this section we will collect, for the reader's convenience, a number of results and definitions that we will use in the following sections. Proofs are omitted or only sketched.

Proposition 1.2. *If $N_{X/\mathbb{P}^6}$ denotes the normal bundle of X in $\mathbb{P}^6$ then the following identities hold:*

$$c_3(N_{X/\mathbb{P}^6}) = d^2 \tag{1}$$

$$c_1(N_{X/\mathbb{P}^6}) = 7H - c_1(X) \tag{2}$$

$$c_2(N_{X/\mathbb{P}^6}) = 21H^2 - c_2(X) - c_1(X) \cdot (7H - c_1(X)) \tag{3}$$

$$\begin{aligned} c_3(N_{X/\mathbb{P}^6}) = 35H^3 + 7H \cdot c_1(X)^2 - 21H^2 \cdot c_1(X) \\ - c_1(X)^3 - 7H \cdot c_2(X) + 2c_1(X) \cdot c_2(X) - c_3(X) \end{aligned} \tag{4}$$

$$\begin{aligned} d^2 - 35d = 7H \cdot c_1(X)^2 - 21H^2 \cdot c_1(X) - c_1(X)^3 - 7H \cdot c_2(X) \\ + 2c_1(X) \cdot c_2(X) - c_3(X). \end{aligned} \tag{5}$$

Proof (Sketch). (1) is a particular case of [10, p. 431]. (2)–(4) are consequence of the standard exact sequence

$$0 \to T_X \to T_{\mathbb{P}^6}|_X \to N_{X/\mathbb{P}^6} \to 0.$$

□

Proposition 1.3. *If E is a rank r vector bundle on an irreducible smooth projective variety B of dimension b, $\mathbb{P}(E)$ the corresponding projectivized bundle and $p : \mathbb{P}(E) \to B$ the canonical projection then*

(i)
$$\sum_{i=0}^{r}(-1)^i p^*(c_i(E)) \cdot h^{r-i} = 0.$$

(ii) *If $r = 2$ and $b = 2$*
$$\begin{aligned} c_1(\mathbb{P}(E)) =& 2h - p^*c_1(E) + p^*c_1(B), \\ c_2(\mathbb{P}(E)) =& 2h \cdot p^*c_1(B) - p^*c_1(E) \cdot p^*c_1(B) + p^*c_2(B), \\ c_3(\mathbb{P}(E)) =& 2h \cdot p^*c_2(B). \end{aligned}$$

(iii) *If $r = 4$ and $b = 1$*
$$\begin{aligned} c_1(\mathbb{P}(E)) =& 4h - p^*c_1(E) + p^*c_1(B), \\ c_2(\mathbb{P}(E)) =& 6h^2 + 4h \cdot p^*c_1(B) - 3h \cdot p^*c_1(E), \\ c_3(\mathbb{P}(E)) =& 4h^3 - 3h^2 \cdot p^*c_1(E) + 6h^2 \cdot p^*c_1(B), \\ c_4(\mathbb{P}(E)) =& 4h^3 \cdot p^*c_1(B). \end{aligned}$$

(iv) *If $r = 3$ and $b = 1$*
$$\begin{aligned} c_1(\mathbb{P}(E)) =& 3h - p^*c_1(E) + p^*c_1(B) \\ c_2(\mathbb{P}(E)) =& 3h^2 + 3h \cdot p^*c_1(B) - 2h \cdot p^*c_1(E) \\ c_3(\mathbb{P}(E)) =& 3h^2 \cdot p^*c_1(B) \end{aligned}$$

where $h \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$.

Proof (Sketch). (i) is [10, p. 429]. ii)–iv) follow from the exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}(E)} \to (p^*E^\vee \otimes \mathcal{O}_{\mathbb{P}(E)}(1)) \to T_{\mathbb{P}(E)} \to p^*T_B \to 0$$
([12, Proposition (17.12), pp. 80–82]). □

Definition 1.4. We call an effective divisor $E \subset X$, an *exceptional plane* if $E \cong \mathbb{P}^2$, $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$ and $\mathcal{O}_X(H) \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^2}(1)$.

Definition 1.5. Consider the pairs (X', H') where X' is a smooth irreducible projective variety and H' is an ample divisor on X', we call (X', H') a *reduction* of (X, H), if there exists a morphism $\sigma : X \to X'$ such that σ is the contraction of all the exceptional planes $E_1, \dots, E_k$ contained in X, and H' is such that $H = \sigma^*(H') - E_1 - \cdots - E_k$.

A standard reference for the notion of reduction is [1].

Proposition 1.6. (i) *The set of exceptional planes of X is finite.*

(ii) *Two exceptional planes of X are disjoint.*

(iii) *If (X', H') is a reduction of (X, H), and $\sigma : X \to X'$ is the corresponding contraction morphism, then X is the blow-up of X' in a finite number of distinct points and σ is the corresponding morphism.*

Proof. [8, p. 102 (11.11)]. □

Notation 1.7. If $\mathcal{S}$ is a set and $h_1 : \mathcal{S} \to \mathbb{N}$, $h_2 : \mathcal{S} \to \mathbb{N}$ two functions we write $h_1 = O(h_2)$ if exists a constant c such that $|h_1(s)| \leq ch_2(s)$, $\forall s \in \mathcal{S}$.

2. Statement and proof of the theorem

Theorem 2.1. *Smooth threefolds in $\mathbb{P}^6$ with ruled surface section, not contained in a fourfold of degree less than or equal to* 15*, are bounded.*

Remark 2.2. Threefolds with ruled surface section are all listed in [1, p. 205, table 7.3] or [11, p. 339, Theorem II]. Since Del Pezzo varieties are classified in [8, p. 71, (8.11)], we have to prove Theorem 2.1 for scrolls in lines, quadric fibrations, Veronese fibrations and scrolls in planes (see [1, 11] for the definitions).

First of all we find a lower bound for the sectional genus of X. This is made by a case by case analysis.

Remark 2.3. If (X, H) is a scroll on a smooth surface then $X \cong \mathbb{P}(E)$, where E is a rank two vector bundle on a smooth projective surface B and $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$.

Notation 2.4. For a scroll in lines $X \subset \mathbb{P}^6$ we will denote by k the number of lines of X that are contained in a general hyperplane section.

Proposition 2.5. *For a scroll in lines in $\mathbb{P}^6$ with ruled surface section*

$$\frac{d^2}{28} + O(d) \leq g.$$

Proof. By Proposition 1.3 (ii) we have

$$\begin{aligned}
c_1(X)^2 &= 4H^2 + p^*c_1(E)^2 + p^*c_1(B)^2 - 4H \cdot p^*c_1(E) \\
&\quad + 4H \cdot p^*c_1(B) - 2p^*c_1(E) \cdot p^*c_1(B), \\
c_1(X)^3 &= 8H^3 + 6H \cdot p^*c_1(E)^2 + 6H \cdot p^*c_1(B)^2 - 12H^2 p^*c_1(E) \\
&\quad + 12H^2 \cdot p^*c_1(B) - 12H \cdot p^*c_1(E) \cdot p^*c_1(B), \\
c_1(X) \cdot c_2(X) &= 4H^2 \cdot p^*c_1(B) - 4H \cdot p^*c_1(E) \cdot p^*c_1(B) \\
&\quad + 2H \cdot p^*c_2(B) + 2H \cdot p^*c_1(B)^2,
\end{aligned}$$

that substituted in (5) yields

$$\begin{aligned} d^2 - 13d = H \cdot p^*c_1(E)^2 + 5H \cdot p^*c_1(B)^2 + 5H^2 \cdot p^*c_1(E) \\ - 11H^2 \cdot p^*c_1(B) - 3Hp^*c_1(E) \cdot p^*c_1(B) - 5H \cdot p^*c_2(B). \end{aligned} \tag{6}$$

Now intersecting the equation in Proposition 1.3 (i) respectively with the cycles $p^*c_1(E)$ and $p^*c_1(B)$ we get

$$\begin{aligned} H \cdot p^*c_1(E)^2 &= H^2 \cdot p^*c_1(E), \\ H \cdot p^*c_1(E) \cdot p^*c_1(B) &= H^2 \cdot p^*c_1(B), \end{aligned} \tag{7}$$

and substituting in (6)

$$d^2 - 13d = 6H^2 \cdot p^*c_1(E) - 14H^2 \cdot p^*c_1(B) + 5K_B^2 - 5H \cdot p^*c_2(B). \tag{8}$$

By the adjunction formula

$$2g - 2 = H^2 \cdot p^*c_1(E) - H^2 \cdot p^*c_1(B)$$

moreover, by [1, p. 282, Theorem (11.1.2)], $d = c_1(E)^2 - c_2(E)$ and $c_2(E) = k$, and then

$$\begin{aligned} H^2 \cdot p^*c_1(E) &= d + k, \\ H^2 \cdot p^*c_1(B) &= d + k - 2g - 2. \end{aligned}$$

Substituting in (8) we get

$$d^2 - 5d + 8k + 28 = 28g + 5K_B^2 - 5c_2(B).$$

To conclude the proof observe that $k \geq 0$ and since B is ruled $K_B^2 - c_2(B) \leq 6$. □

Remark 2.6. Let X be a quadric fibration, X is a smooth divisor in $\mathbb{P}(E)$, projectivized of a vector bundle E, $\mathrm{rk}(E) = 4$, on B a smooth curve. If h denotes the class of $\mathcal{O}_{\mathbb{P}(E)}(1)$ in $\mathrm{Pic}(X)$ then $X \in |2h - \pi^*L|$, where L is a divisor on B, and $\pi : \mathbb{P}(E) \to B$ is the canonical projection associated $\mathbb{P}(E)$.

$$N_{X/\mathbb{P}(E)} = (2h - \pi^*L)_{|X} = 2h_{|X} - (\pi^*L)_{|X} = 2H - p^*L$$

where $H = h_{|X}$, and p denotes the restriction of π to X. H embeds X in $\mathbb{P}^6$.

Proposition 2.7. *For a quadric fibration in* $\mathbb{P}^6$

$$\frac{3d^2}{70} + O(d) \leq g.$$

Proof. From the exact sequence

$$0 \to T_X \to T_{\mathbb{P}(E)}|_X \to N_{X/\mathbb{P}(E)} \to 0,$$

we have

$$\begin{aligned} c_1(X) &= c_1(T_{\mathbb{P}(E)|X}) - 2H + p^*L, \\ c_2(X) &= c_2(T_{\mathbb{P}(E)}|X) - 2H \cdot c_1(X) + c_1(X) \cdot p^*L, \\ c_3(X) &= c_3(T_{\mathbb{P}(E)}|X) - 2H \cdot c_2(X) + c_2(X) \cdot p^*L. \end{aligned}$$

But by Proposition 1.3 (iii)

$$\begin{aligned} c_1(T_{\mathbb{P}(E)|X}) &= 4H - p^*c_1(E) + p^*c_1(B), \\ c_2(T_{\mathbb{P}(E)|X}) &= 6H^2 + 4H \cdot p^*c_1(B) - 3H \cdot p^*c_1(E), \\ c_3(T_{\mathbb{P}(E)|X}) &= 4H^3 - 3H^2 \cdot p^*c_1(E) + 6H^2 \cdot p^*c_1(B), \end{aligned}$$

hence

$$\begin{aligned} c_1(X) &= 2H - p^*c_1(E) + p^*c_1(B) + p^*L, \\ c_2(X) &= 2H^2 - H \cdot p^*c_1(E) + 2H \cdot p^*c_1(B), \\ c_3(X) &= -H^2 \cdot p^*c_1(E) + 2H^2 \cdot p^*c_1(B) + 2H^2 \cdot p^*L. \end{aligned}$$

Substituting in (5), and noting that $-H^2 \cdot p^*c_1(B) = 4g(B) - 4$ (where $g(B)$ denote the geometric genus of B), we get

$$d^2 - 7d = 5H^2 \cdot p^*c_1(E) + 36g(B) - 36 - 3H^2 \cdot p^*L. \tag{9}$$

Denote by C the general curve section of X, then we have a 2:1 map $p_{|C} : C \to B$. By adjunction formula

$$K_C = H^2 \cdot p^*c_1(E) - H^2 \cdot p^*c_1(B) - H^2 \cdot p^*L,$$

substituting in

$$K_C = (p_{|C})^*(K_B) + R,$$

(where as usual R denotes the ramification divisor of $p_{|C}$) and using Hurwitz's formula for the degree of R we have

$$2(g+1) - 4g(B) = H^2 \cdot p^*c_1(E) - H^2 \cdot p^*L.$$

In $\mathbb{C}(E)$ we have $X \in |2h - \pi^*L|$, and, by Proposition 1.3 (i), $h^4 = h^3 \cdot \pi^*c_1(E)$ and

$$2d = 2H^2 \cdot p^*c_1(E) - H^2 \cdot p^*L. \tag{10}$$

The two equalities above imply

$$c_1(E) = d - (g+1) + 2g(B),$$
$$\deg(L) = d - 2(g+1) + 4g(B),$$

and substituting in (9) we get

$$d^2 - 11d + 36 = 2(g+1) + 32g(B). \tag{11}$$

$N_{X/\mathbb{P}^6}(-1)$ is generated by its sections. Then, by the positivity results of Fulton and Lazarsfeld ([9, chapter 12]), we have that the second Segre class of $N_{X/\mathbb{P}^6}(-1)$ satisfies the following inequality

$$H \cdot s_2(N_{X/\mathbb{P}^6}(-1)) \geq 0.$$

By the same type of computations as before, writing this inequality for a quadric fibration we obtain

$$12g(B) \leq 8g + 4 \tag{12}$$

that with (11) proves the Proposition. □

Remark 2.8. Let X be a Veronese fibration, then exists a reduction σ of (X, H) to (X', H'), where $\sigma : X \to X'$ is the blow-up in k distinct points (Proposition 1.6), X' is the projectivized bundle of a rank 3 vector bundle on B, a smooth curve. $H' = 2h - p^*L$, $H = \sigma^*(H') - \sum_{i=1}^{k} E_i$, where p is the canonical projection on the base, $h \in |\mathcal{O}_{X'}(1)|$, E_i are the exceptional divisors of σ and L is a divisor on B.

Proposition 2.9. *For a Veronese fibration in* $\mathbb{P}^6$

$$\frac{d^2}{24} + O(d) \leq g.$$

Proof. Denote by $c(X)$ and $c(X')$ the total Chern class respectively of X and X'. By [9, pp. 300–301, Theorem (15.4) and Example (15.4.2)],

$$c(X) = \sigma^* c(X') + \sum_{i=1}^{k} \{-2E_i + 2E_i^3\}. \tag{13}$$

Since X and X' are smooth, $\sigma^* : A^*X' \to A^*X$ is a morphism of graded rings ([9, p. 140]), and since $\mathcal{O}_{E_i}(E_i) \cong \mathcal{O}(-1)$ then $E_i^3 = 1$. From (13), recalling that E_i

are disjoint by Proposition 1.6 we obtain

$$
\begin{gathered}
H = \sigma^*(H') - \sum_{i=1}^{k} E_i, \\
H^2 = \sigma^*(H')^2 + \sum_{i=1}^{k} E_i^2, \\
c_1(X) = \sigma^* c_1(X') - 2\sum_{i=1}^{k} E_i, \\
c_1(X)^2 = \sigma^* c_1(X')^2 + 4\sum_{i=1}^{k} E_i^2, \\
c_1(X)^3 = \sigma^* c_1(X')^3 - 8k, \\
c_2(X) = \sigma^* c_2(X'), \\
c_3(X) = \sigma^* c_3(X') + 2k,
\end{gathered}
$$

and substituting in (5)

$$
\begin{aligned}
d^2 - 35d = \sigma^*(7H' \cdot c_1(X')^2 - 21(H')^2 \cdot c_1(X') - c_1(X')^3 \\
- 7H' \cdot c_2(X') + 2c_1(X') \cdot c_2(X') - c_3(X')) + 20k. \quad (14)
\end{aligned}
$$

As in the preceding cases, by Proposition 1.3 (i) and (iv), we can write (14) as

$$
d^2 - 35d = -\frac{35}{2}(d+k) - 48(2 - 2g(B)) + 20k. \quad (15)
$$

By adjunction formula

$$
k = 2g - d - 8g(B) + 6
$$

that with (15) proves the Proposition. □

As above by Chern classes computations we have the following:

Proposition 2.10. *For a scroll in planes in* $\mathbb{P}^6$

$$
\frac{d^2}{12} + O(d) \leq g.
$$

We can now conclude.

Proof of Theorem 2.1. We may suppose our threefold X non degenerate. Indeed, if $X \subset \mathbb{P}^5$ by [2, Theorem 2] X would have bounded degree. It follows that the general curve section of X, $C \subset \mathbb{P}^4$, is non-degenerate. Suppose that X is not contained in a fourfold of degree less than 15. If d is large enough then C in not contained in a surface of $\mathbb{P}^4$ of degree less than 15. If it were so then by [3, p. 97, Theorem (0.2)], X would be contained in a fourfold of degree less than 15, that is impossible by our assumption.

By [4, Main Theorem], if d is large enough,

$$g \leq \frac{d^2}{30} + O(d),$$

which by Propositions 2.5–2.10 concludes the proof. □

3. Boundedness for threefolds in a fourfold of bounded degree

Having in mind Theorem 2.1, one may ask, if it is possible to prove a full boundedness result by an argument as [7, Lemma 1, p. 3]. We have not been able to prove such a general statement so far, but only a partial one as in [2, Proposition 5.1, p. 331].

Let then X be a threefold in $\mathbb{P}^6$ with ruled hyperplane section such that $X \subset W \subset \mathbb{P}^6$ where $\dim(W) = 4$ and $\deg(W) = \sigma < 15$ fixed. By choosing two hypersurfaces of $\mathbb{P}^6$ containing W hence X, we can define a map of vector bundles on X

$$\mu : N_{X/\mathbb{P}^6} \to \mathcal{O}_X(\sigma)^{\oplus 2}.$$

Following [9] we will denote by $D_1(\mu)$ the scheme where μ has rank less than or equal to one. First of all let us make a remark.

Remark 3.1. If X is a scroll in lines, quadric fibration, Veronese fibration or scroll in planes and S denote its generic hyperplane section then by a Chern classes computation and Riemann-Roch we have

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S).$$

Notations are as in section 2. For a scroll in lines, in view of (7),

$$\begin{aligned}
24\chi(\mathcal{O}_X) &= c_1(X) \cdot c_2(X) \\
&= 4H^2 \cdot p^*c_1(B) - 4H \cdot p^*c_1(E) \cdot p^*c_1(B) + 2H \cdot p^*c_2(B) + 2H \cdot p^*c_1(B)^2 \\
&= 4H^2 \cdot p^*c_1(B) - 4H^2 \cdot p^*c_1(B) + 2H \cdot^* c_2(B) + 2H \cdot p^*c_1(B)^2 \\
&= 2c_2(B) + 2K_B^2 = 24\chi(\mathcal{O}_B),
\end{aligned}$$

since χ is a birational invariant this is our claim. For a quadric fibration and a Veronese fibration we have respectively

$$\begin{aligned}
24\chi(\mathcal{O}_X) &= c_1(X) \cdot c_2(X) \\
&= 4H^3 - 4H^2 \cdot p^*c_1(E) + 6H^2 \cdot p^*c_1(B) + 2H^2 \cdot p^*L \\
&= 6H^2 \cdot p^*c_1(B) = 24(1 - g(B)), \qquad \text{(by (10))}
\end{aligned}$$

$$\begin{aligned}24\chi(\mathcal{O}_X) &= c_1(X')\cdot c_2(X')\\ &= 9h^3 - 9h^2\cdot p^*c_1(E) + 12h^2\cdot p^*c_1(B)\\ &= 12h^2\cdot p^*c_1(B) = 24(1-g(B)). \qquad \text{(by Proposition 1.3 (i))}\end{aligned}$$

The claim follows in this cases noting that S is a birationally ruled surface over a curve of genus $g(B)$. As above by a Chern classes computation one can prove the claim for scroll in planes.

Proposition 3.2. *Threefolds in $\mathbb{P}^6$ (with ruled surface section) contained in a fourfold of bounded degree σ such that $D_1(\mu)$, for a map μ constructed as above, has pure codimension two, are bounded.*

Proof. To make the proof clearer we split it in two steps. In Step 1 we prove an upper bound for the sectional genus of X. Such bound, by the results in section 2, proves the Proposition provided that the degree of the fourfold is not equal to three. In step 2 we prove the Proposition in this particular case by a direct computation. Notations are as in section 2.

Step 1. *For a fixed degree $\sigma \geq 3$ we have*

$$g \leq \frac{d^2}{28(\sigma-2)} + O(d), \tag{16}$$

for a scroll in lines, Veronese fibration, scroll in planes while

$$g \leq \frac{3d^2}{84(\sigma-2)-14} + O(d), \tag{17}$$

for a quadric fibration.

Define

$$\gamma = c_1(\mathcal{O}_X(\sigma)^{\oplus 2} - N_{X/\mathbb{P}^6})^2 - c_2(\mathcal{O}_X(\sigma)^{\oplus 2} - N_{X/\mathbb{P}^6}) \in A_1X$$

where AX denotes the Chow ring of X. Since by hypothesis $D_1(\mu)$ has pure codimension 2 by [9, p. 254, Theorem 14.4(d)], we have $\gamma = [D_1(\mu)]$ in A_1X, where $[D_1(\mu)]$ denotes the cycle associated to $D_1(\mu)$ in A_1X, hence $0 \leq \gamma\cdot H$.

By Proposition 1.2 we can write

$$\gamma = c_2(N_{X/\mathbb{P}^6}) + 3\sigma^2H^2 - 2\sigma H\cdot(7H+K_X),$$

and

$$0 \leq H\cdot c_2(N_{X/\mathbb{P}^6}) + 3\sigma^2H^3 - 2\sigma H^2\cdot(7H+K_X). \tag{18}$$

By (3), intersecting with H,

$$H\cdot c_2(N_{X/\mathbb{P}^6}) = 21H^3 - H\cdot c_2(X) - H\cdot c_1(X)\cdot(7H - c_1(X)).$$

Solving (5) with respect to $H \cdot c_2(X)$ and substituting in the preceding equality we get

$$H \cdot c_2(N_{X/\mathbb{P}^6}) = \frac{d^2}{7} + 16d + 4H^2 \cdot K_X - \frac{K_X^3}{7} - \frac{48}{7}\chi(\mathcal{O}_X) + \frac{c_3(X)}{7}. \qquad (19)$$

In view of (19) and Remark 3.1, the inequality (18) becomes

$$0 \le \frac{d^2}{7} + (3\sigma^2 - 10\sigma + 8)d + 4(2-\sigma)(g-1) + \Delta$$

where we set

$$\Delta = -\frac{K_X^3}{7} - \frac{48}{7}\chi(\mathcal{O}_S) + \frac{c_3(X)}{7}.$$

To find an upper bound for g, it is sufficient to find one for Δ. We proceed by a case by case analysis as in section 2. For a scroll in lines we have

$$\begin{aligned} -K_X^3 &= 8H^3 - 6H^2 \cdot p^*c_1(E) + 6H \cdot p^*c_1(B)^2 \\ &= 6H \cdot p^*c_1(B)^2 + 2d = 6K_B^2 + 2d - 6k, \end{aligned}$$

$$48\chi(\mathcal{O}_S) = 48\chi(\mathcal{O}_B) = 4(K_B^2 + c_2(B)),$$

$$c_3(X) = 2c_2(B),$$

hence

$$\Delta = \frac{2}{7}(d + K_B^2 - c_2(B) - 3k) \le \frac{2}{7}(d+6)$$

since B is ruled. This proves Step 1 for a scroll in lines.

For a quadric fibration

$$\begin{aligned} -K_X^3 &= -4H^2 \cdot p^*c_1(E) + 12H^2 \cdot p^*c_1(B) + 8H^2 \cdot p^*L \\ \chi(\mathcal{O}_S) &= 1 - g(B), \\ c_3(X) &= -H^2 \cdot p^*c_1(E) + 2H^2 \cdot p^*c_1(B) + 2H^2 \cdot p^*L. \end{aligned}$$

Then, by definition of Δ, (9), and (12), we have

$$\Delta \le \frac{2}{3}g + \frac{10}{7}d - \frac{2}{3}$$

The same kind of argument proves that Δ is less than or equal to a constant for a Veronese fibration and a scroll in planes. This concludes the proof of Step 1.

By (16) and (17) and by Propositions 2.5–2.10, to conclude the proof of the Proposition it is sufficient to prove

Step 2. *Scrolls in lines and quadric fibrations contained in a fourfold $W \subseteq \mathbb{P}^6$ of degree $\sigma = 3$ are bounded.*

In this case W is a variety of minimal degree. By the classification of such varieties, see for example [6], W is a cone over a rational normal scroll in $\mathbb{P}^5$ and, (adopting notations of [6]) W is projectively equivalent to $S(0,1,1,1)$.

Set $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}$, then the linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ defines a birational morphism $\mathbb{P}(\mathcal{E}) \to W$. Denote by $\tilde{X}$ the strict transform of X by this morphism. Since X is smooth, $\tilde{X}$ is smooth too.

It is well known that $\operatorname{Pic}(\mathbb{P}(\mathcal{E})) = \mathbb{Z}h + \mathbb{Z}f$, where f is a class of a fibre of the canonical projection $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$, and $h \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. Write $\tilde{X} = ah + bf$, $\tilde{H} = h_{|\tilde{X}}$, and $\tilde{F} = f_{|\tilde{X}}$. Then

$$\tilde{H}^3 = \deg(X) = d = 3a + b,$$
$$\tilde{H}^2 \cdot \tilde{F} = a.$$

Since the general curve section of $\tilde{X}$ by $\tilde{H}$ has genus equal to g, the sectional genus of X, then by the following exact sequence

$$0 \to T_{\tilde{X}} \to T_{\mathbb{P}(\mathcal{E})}|_{\tilde{X}} \to N_{\tilde{X}/\mathbb{P}(\mathcal{E})} \to 0,$$

and adjunction formula

$$2g - 2 = (1+b)a + (a-2)(3a+b).$$

Moreover denote by $\tilde{S}$ the general surface section of $\tilde{X}$ by $\tilde{H}$, then by the exact sequence

$$0 \to T_{\tilde{S}} \to T_{\tilde{X}}|_{\tilde{S}} \to N_{\tilde{S}/\tilde{X}} \to 0,$$

we are able to compute

$$c_1(\tilde{S}) = ((3-a)\tilde{H} - (b+1)\tilde{F})_{|\tilde{S}} \tag{20}$$
$$c_2(\tilde{S}) = \big((a^2 - 3a + 3)\cdot \tilde{H}^2 + (2ab + a - 3b)\tilde{H}\cdot\tilde{F}\big)_{|\tilde{S}} \tag{21}$$

Now observe that $d \geq 0$, $K^2_{\tilde{S}} \leq 9$, and since χ is a birational invariant, by Remark 3.1 we have that $\chi(\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_{\tilde{S}})$. In view of (20) and (21) we can rewrite the terms in this equality and in the two inequalities above as functions of a and b obtaining

$$3a + b \geq 0,$$
$$b(3a^2 - 12a + 9) \leq -3a^3 + 16a^2 - 21a + 9,$$
$$b(12a^2 - 8a) = 3a^4 - 10a^3 + 13a^2 - 2a.$$

The above inequalities imply that a is bounded. Again writing the terms in the inequality of Proposition 2.5 and of Proposition 2.7 as functions of a and b we obtain

$$b^2 + \phi(a,b) \leq 0, \tag{22}$$

where $\phi(a,b)$ is a polynomial in a and b of degree one in b. Since a is bounded, by (22) b is also bounded, hence d is bounded. □

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